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Extension of noncommutative soliton hierarchies

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Abstract

A linear system, which generates a Moyal-deformed two-dimensional soliton equation as an integrability condition, can be extended to a three-dimensional linear system, treating the deformation parameter as an additional coordinate. The supplementary integrability conditions result in a first-order differential equation with respect to the deformation parameter, the flow of which commutes with the flow of the deformed soliton equation. In this way, a deformed soliton hierarchy can be extended to a bigger hierarchy by including the corresponding deformation equations. We prove the extended hierarchy properties for the deformed AKNS hierarchy, and specialize to the cases of deformed NLS, KdV and mKdV hierarchies. Corresponding results are also obtained for the deformed KP hierarchy. A deformation equation determines a kind of Seiberg–Witten map from classical solutions to solutions of the respective ‘noncommutative’ deformed equation.

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1. Introduction

Because of their appearance (see [1, 2], in particular) in certain limits of string theories (see [3] for a review), noncommutative generalizations of soliton equations, obtained from classical models essentially by replacing the ordinary commutative product of functions with a noncommutative Moyal \ast -product, have attracted considerable interest recently [4–21]. One can say that such equations live on a ‘noncommutative space’, they are not simply ordinary evolution equations of non-commuting objects such as matrices. We will use ‘noncommutative’ (nc) in this sense.

In previous work [5, 9] on noncommutative versions of certain soliton equations, we were able to find a differential equation of first order in the deformation parameter θ , which allowed us to calculate solutions of the noncommutative soliton equation order by order in θ from

solutions of the corresponding classical soliton equation. Thus, in these cases there is a map from solutions of the ‘commutative’ to solutions of the ‘noncommutative’ equation which is analogous to the Seiberg–Witten (SW) *map* of [22]. Because of this reason we call such a θ -evolution equation a *SW deformation equation*.

A systematic way of generating such an equation will be described now. We consider a nonlinear equation (or a system of equations) which is integrable in the sense that it can be expressed as the integrability condition of a linear system of equations. Prominent examples are given by soliton equations (see [23], for example). Let us look at a linear system of the form

$$\partial_x \psi = \mathcal{U} * \psi \quad \partial_t \psi = \mathcal{V} * \psi \quad (1.1)$$

where \mathcal{U}, \mathcal{V} are $N \times N$ matrices of functions (or suitable operators). The associative noncommutative $*$ -product is defined by

$$f * g = \mathbf{m} \circ e^{\theta P/2}(f \otimes g) \quad P = \partial_t \otimes \partial_x - \partial_x \otimes \partial_t \quad (1.2)$$

using $\mathbf{m}(f \otimes g) = fg$, for functions f, g of the coordinates x, t and the deformation parameter θ . The integrability condition of the above system is

$$\partial_t \mathcal{U} - \partial_x \mathcal{V} + [\mathcal{U}, \mathcal{V}]_* = 0 \quad (1.3)$$

which is a noncommutative version of the familiar zero curvature condition [23]. We have set $[\mathcal{U}, \mathcal{V}]_* = \mathcal{U} * \mathcal{V} - \mathcal{V} * \mathcal{U}$. The crucial step now is to assume that there is also a (compatible) linear equation for the parameter θ , i.e.

$$\partial_\theta \psi = \mathcal{W} * \psi \quad (1.4)$$

where \mathcal{W} is an $N \times N$ matrix of functions (or operators). This gives rise to further integrability conditions. Using the identity

$$\partial_\theta (f * g) = (\partial_\theta f) * g + f * (\partial_\theta g) + \frac{1}{2}((\partial_t f) * (\partial_x g) - (\partial_x f) * (\partial_t g)) \quad (1.5)$$

we find the following equations:

$$\partial_\theta \mathcal{U} = \partial_x \mathcal{W} + [\mathcal{W}, \mathcal{U}]_* + \frac{1}{2}[(\partial_x \mathcal{U}) * \mathcal{V} - (\partial_t \mathcal{U}) * \mathcal{U}] \quad (1.6)$$

$$\partial_\theta \mathcal{V} = \partial_t \mathcal{W} + [\mathcal{W}, \mathcal{V}]_* + \frac{1}{2}[(\partial_x \mathcal{V}) * \mathcal{V} - (\partial_t \mathcal{V}) * \mathcal{U}]. \quad (1.7)$$

These are zero curvature conditions such as (1.3), but modified because of the θ -dependence of the $*$ -product. It is this modification which leads to new structures. Because of the explicit dependence of the right-hand sides of (1.6) and (1.7) on θ , through the use of the $*$ -product, these equations are non-autonomous.

Suppose we have achieved in formulating a deformed soliton equation for a variable u , on which \mathcal{U} and \mathcal{V} depend, in the form (1.3). If, as a consequence of the specific form of our choices of \mathcal{U} and \mathcal{V} , the last two equations reduce to a single (deformation) equation of the form $\partial_\theta u = F$ where F stands for an expression in u and its t - and x -derivatives, but no θ -derivatives, then $\partial_\theta u = F$ is automatically consistent with the deformed soliton equation. This means that the flows defined by the soliton equation and the deformation equation commute.

This takes us to another point of view. A characteristic feature of classical soliton equations is the existence of infinitely many symmetries. A symmetry is given by an evolution equation which commutes with the soliton equation (see [24], for example). It is an evolution equation with respect to a new coordinate, say θ , and one treats the soliton field, say u , now as a function of x, t, θ . What we are doing is very similar, but differs in so far that we allow for an explicit dependence of the soliton equation on this new coordinate θ . We restrict this explicit dependence on θ to the $*$ -product, however.

The symmetries of classical soliton equations lead to hierarchies. These are families of commuting evolution equations. They are easily (Moyal-) deformed to hierarchies of noncommutative equations. But then we can ask for further symmetries in the sense explained above. In sections 2 and 3, we elaborate this problem for the case of the AKNS hierarchy (see [25–27], for example, and the references given there), which admits reductions to the NLS, KdV and mKdV hierarchies. It turns out that the noncommutative AKNS hierarchy can be enlarged to a much bigger hierarchy which includes a new hierarchy of deformation equations. The corresponding mathematical framework is developed in section 2. Section 3 then specializes this framework a bit. In particular, it provides explicit formulae for the first members of the noncommutative versions of the NLS, KdV and mKdV hierarchies, including their SW deformation hierarchies.

In section 4, we derive an extension of the noncommutative version [16] of the KP hierarchy [28–30]. Section 5 contains some conclusions.

2. A hierarchy of Lax equations and associated SW deformation equations

Let us consider a family of Lax equations [31, 32]

$$\partial_{t_n} \mathcal{V} = [\mathcal{V}^{(n)}, \mathcal{V}]_* \quad n = 0, 1, \dots \tag{2.1}$$

with³

$$\mathcal{V} = \sum_{k=0}^{\infty} \lambda^{-k} V_k \quad \mathcal{V}^{(n)} = (\lambda^n \mathcal{V})_{\geq 0} \tag{2.2}$$

where the V_k do not depend on the (formal) parameter λ and the last expression means taking the part of $\lambda^n \mathcal{V}$ which corresponds to non-negative powers of λ , hence $(\lambda^n \mathcal{V})_{\geq 0} = \sum_{k=0}^n \lambda^{n-k} V_k$. Then

$$\bar{\mathcal{V}}^{(n)} = (\lambda^n \mathcal{V})_{< 0} = \lambda^n \mathcal{V} - \mathcal{V}^{(n)} = \sum_{k=1}^{\infty} \lambda^{-k} V_{n+k} \tag{2.3}$$

only contains terms with negative powers of λ . An example of the structure (2.1) (without the deformation) appeared in [26] as a formulation of the AKNS hierarchy. The $*$ -product of two functions of the variables t_0, t_1, \dots is now defined by

$$f * g = \mathbf{m} \circ \exp \left(\sum_{m,n} \theta_{m,n} P_{m,n} / 4 \right) (f \otimes g) \quad P_{m,n} = \partial_{t_m} \otimes \partial_{t_n} - \partial_{t_n} \otimes \partial_{t_m} \tag{2.4}$$

where $\theta_{m,n} = -\theta_{n,m}$ are deformation parameters. Using

$$[\mathcal{V}^{(n)}, \mathcal{V}]_* = \sum_{j=1}^{\infty} \lambda^{-j} \sum_{k=0}^{\min\{j-1, n\}} [V_k, V_{n+j-k}]_* \tag{2.5}$$

(2.1) turns out to be equivalent to

$$\partial_{t_n} V_0 = 0 \quad \partial_{t_n} V_j = \sum_{k=0}^{\min\{j-1, n\}} [V_k, V_{n+j-k}]_* \quad j = 1, 2, \dots \tag{2.6}$$

We recall an important result of soliton analysis (see [26, 31, 32], for example).

Theorem 1. *The Lax equations (2.1) imply*

$$\partial_{t_n} \mathcal{V}^{(m)} - \partial_{t_m} \mathcal{V}^{(n)} + [\mathcal{V}^{(m)}, \mathcal{V}^{(n)}]_* = 0 \tag{2.7}$$

for all $m, n \geq 0$. The flows (2.1) commute with each other.

³ We are dealing with formal series in λ , no convergence is required.

Proof. We consider two of equations (2.1) corresponding to different non-negative integers n and m . Multiplying the first by λ^m , and decomposing it into non-negative and negative parts (with respect to powers of λ) we find

$$\partial_{t_n} \mathcal{V}^{(m)} + [\mathcal{V}^{(m)}, \mathcal{V}^{(n)}]_* + \partial_{t_n} \bar{\mathcal{V}}^{(m)} - [\mathcal{V}^{(n)}, \bar{\mathcal{V}}^{(m)}]_* = 0.$$

In the same way we obtain

$$\partial_{t_m} \mathcal{V}^{(n)} + [\mathcal{V}^{(n)}, \mathcal{V}^{(m)}]_* + \partial_{t_m} \bar{\mathcal{V}}^{(n)} - [\mathcal{V}^{(m)}, \bar{\mathcal{V}}^{(n)}]_* = 0.$$

Taking the difference of both equations and using

$$0 = [\lambda^n \mathcal{V}, \lambda^m \mathcal{V}]_* = [\mathcal{V}^{(n)}, \mathcal{V}^{(m)}]_* + [\mathcal{V}^{(n)}, \bar{\mathcal{V}}^{(m)}]_* - [\mathcal{V}^{(m)}, \bar{\mathcal{V}}^{(n)}]_* + [\bar{\mathcal{V}}^{(n)}, \bar{\mathcal{V}}^{(m)}]_*$$

yields

$$\partial_{t_n} \mathcal{V}^{(m)} - \partial_{t_m} \mathcal{V}^{(n)} + [\mathcal{V}^{(m)}, \mathcal{V}^{(n)}]_* = \partial_{t_m} \bar{\mathcal{V}}^{(n)} - \partial_{t_n} \bar{\mathcal{V}}^{(m)} - [\bar{\mathcal{V}}^{(n)}, \bar{\mathcal{V}}^{(m)}]_*.$$

Since both sides of this equation live in different spaces, we conclude that (2.7) holds⁴. Using the Jacobi identity, we find

$$(\partial_{t_n} \partial_{t_m} - \partial_{t_m} \partial_{t_n}) \mathcal{V} = [\partial_{t_n} \mathcal{V}^{(m)} - \partial_{t_m} \mathcal{V}^{(n)} + [\mathcal{V}^{(m)}, \mathcal{V}^{(n)}]_*, \mathcal{V}]_* = 0$$

which expresses the commutativity of the flows. \square

Theorem 1 justifies calling (2.1) a *hierarchy* of Lax equations. For fixed n , equation (2.1) is the compatibility condition of the linear system⁵

$$\partial_{t_n} \psi = \mathcal{V}^{(n)} * \psi \quad \mathcal{V} * \psi = \lambda \psi. \quad (2.8)$$

The integrability condition for the two members of the linear system, corresponding to evolution parameters t_m and t_n , is (2.7) which we have shown to be a consequence of (2.1). Following the recipe of the introduction, we extend the family of linear systems with

$$\partial_{\theta_{m,n}} \psi = \mathcal{W}^{(m,n)} * \psi \quad m, n = 0, 1, \dots \quad (2.9)$$

where $\mathcal{W}^{(m,n)} = -\mathcal{W}^{(n,m)}$. The identity (1.5) now takes the form

$$\partial_{\theta_{m,n}} (f * g) = (\partial_{\theta_{m,n}} f) * g + f * (\partial_{\theta_{m,n}} g) + \frac{1}{2} ((\partial_{t_m} f) * (\partial_{t_n} g) - (\partial_{t_n} f) * (\partial_{t_m} g)). \quad (2.10)$$

With its help we obtain the additional integrability conditions

$$\partial_{t_n} \mathcal{W}^{(k,l)} - \partial_{\theta_{k,l}} \mathcal{V}^{(n)} = [\mathcal{V}^{(n)}, \mathcal{W}^{(k,l)}]_* + \frac{1}{2} (\partial_{t_k} \mathcal{V}^{(n)} * \mathcal{V}^{(l)} - \partial_{t_l} \mathcal{V}^{(n)} * \mathcal{V}^{(k)}) \quad (2.11)$$

$$\partial_{\theta_{m,n}} \mathcal{V} = [\mathcal{W}^{(m,n)}, \mathcal{V}]_* - \frac{1}{2} (\partial_{t_m} \mathcal{V} * \mathcal{V}^{(n)} - \partial_{t_n} \mathcal{V} * \mathcal{V}^{(m)}) \quad (2.12)$$

and

$$0 = \partial_{\theta_{k,l}} \mathcal{W}^{(m,n)} - \partial_{\theta_{m,n}} \mathcal{W}^{(k,l)} + [\mathcal{W}^{(m,n)}, \mathcal{W}^{(k,l)}]_* + \frac{1}{2} (\partial_{t_k} \mathcal{W}^{(m,n)} * \mathcal{V}^{(l)} - \partial_{t_l} \mathcal{W}^{(m,n)} * \mathcal{V}^{(k)} - \partial_{t_m} \mathcal{W}^{(k,l)} * \mathcal{V}^{(n)} + \partial_{t_n} \mathcal{W}^{(k,l)} * \mathcal{V}^{(m)}). \quad (2.13)$$

The above linear system was mainly needed to find the SW deformation equation (2.12). If (2.11) holds, it follows that the Lax flows (2.1) commute with the flows given by (2.12). Moreover, if (2.13) holds, the deformation flows (2.12) commute. After determining $\mathcal{W}^{(m,n)}$, we show that these conditions are indeed satisfied as a consequence of (2.1) and (2.12).

⁴ Writing (2.1) in the equivalent form $\partial_{t_n} \mathcal{V} = -[\bar{\mathcal{V}}^{(n)}, \mathcal{V}]_*$, we see that $\partial_{t_n} \bar{\mathcal{V}}^{(m)} - \partial_{t_m} \bar{\mathcal{V}}^{(n)} - [\bar{\mathcal{V}}^{(m)}, \bar{\mathcal{V}}^{(n)}]_* = 0$ is equivalent to (2.7).

⁵ We assume that λ is not only independent of the coordinates t_n , but also does not depend on the deformation parameters.

V_0 does not depend on the coordinates t_n and we assume that it also does not depend on the deformation parameters $\theta_{m,n}$. As a consequence $(\partial_{\theta_{m,n}} \mathcal{V})_{\geq 0} = 0$, and since

$$((\partial_{t_m} \mathcal{V}) * \bar{\mathcal{V}}^{(n)})_{\geq 0} = 0 = ((\partial_{t_m} \mathcal{V}) * \bar{\mathcal{V}}^{(m)})_{\geq 0} \tag{2.14}$$

equation (2.12) implies

$$\begin{aligned} ([\mathcal{W}^{(m,n)}, \mathcal{V}]_*)_{\geq 0} &= \frac{1}{2}((\partial_{t_m} \mathcal{V}) * \mathcal{V}^{(n)} - (\partial_{t_m} \mathcal{V}) * \mathcal{V}^{(m)})_{\geq 0} \\ &= \frac{1}{2}((\partial_{t_m} \lambda^n \mathcal{V}) * \mathcal{V} - (\partial_{t_m} \lambda^m \mathcal{V}) * \mathcal{V})_{\geq 0} \\ &= \frac{1}{2}((\partial_{t_m} \mathcal{V}^{(n)}) * \mathcal{V} - (\partial_{t_m} \mathcal{V}^{(m)}) * \mathcal{V})_{\geq 0} \\ &= \frac{1}{2}([\mathcal{V}^{(m)}, \mathcal{V}^{(n)}]_* * \mathcal{V})_{\geq 0}. \end{aligned} \tag{2.15}$$

In the last step we used (2.7). The last expression is further evaluated as follows,

$$\begin{aligned} ([\mathcal{V}^{(m)}, \mathcal{V}^{(n)}]_* * \mathcal{V})_{\geq 0} &= ([\mathcal{V}^{(m)}, \lambda^n \mathcal{V} - \bar{\mathcal{V}}^{(n)}]_* * \mathcal{V})_{\geq 0} \\ &= ([\mathcal{V}^{(m)} * (\lambda^n \mathcal{V}), \mathcal{V}]_* - [\lambda^m \mathcal{V} - \bar{\mathcal{V}}^{(m)}, \bar{\mathcal{V}}^{(n)}]_* * \mathcal{V})_{\geq 0} \\ &= ([\mathcal{V}^{(m)} * (\lambda^n \mathcal{V}), \mathcal{V}]_* + [\bar{\mathcal{V}}^{(n)} * (\lambda^m \mathcal{V}), \mathcal{V}]_*)_{\geq 0} \\ &= (-[\bar{\mathcal{V}}^{(m)} * (\lambda^n \mathcal{V}), \mathcal{V}]_* + [\bar{\mathcal{V}}^{(n)} * (\lambda^m \mathcal{V}), \mathcal{V}]_*)_{\geq 0} \end{aligned} \tag{2.16}$$

since $[(\lambda^m \mathcal{V}) * (\lambda^n \mathcal{V}), \mathcal{V}] = 0$. Hence

$$([\mathcal{W}^{(m,n)}, \mathcal{V}]_*)_{\geq 0} = \frac{1}{2}([\bar{\mathcal{V}}^{(n)} * \mathcal{V}^{(m)} - \bar{\mathcal{V}}^{(m)} * \mathcal{V}^{(n)}, \mathcal{V}]_*)_{\geq 0}. \tag{2.17}$$

This suggests to choose

$$\mathcal{W}^{(m,n)} = \frac{1}{2}(\bar{\mathcal{V}}^{(n)} * \mathcal{V}^{(m)} - \bar{\mathcal{V}}^{(m)} * \mathcal{V}^{(n)})_{\geq 0} \tag{2.18}$$

which solves the ‘non-negative’ part of (2.12). Inserting the expansions for the $\mathcal{V}^{(n)}$ and the $\bar{\mathcal{V}}^{(n)}$ in powers of λ , we find

$$\bar{\mathcal{V}}^{(n)} * \mathcal{V}^{(m)} = \sum_{-m+1}^{\infty} \lambda^{-j} \sum_{k=0}^{\min\{m-1+j, m\}} V_{m+n+j-k} * V_k \tag{2.19}$$

and thus

$$\mathcal{W}^{(m,n)} = -\frac{1}{2} \sum_{j=0}^{n-1} \lambda^j \sum_{k=1}^{\min\{n-j, n-m\}} V_{m+k} * V_{n-j-k} \quad \text{if } m < n. \tag{2.20}$$

In order to elaborate the SW deformation equations (2.12), it is convenient to insert (2.18) and to use (2.1) to rewrite them in the form

$$\partial_{\theta_{m,n}} \mathcal{V} = [\mathcal{V}, \bar{\mathcal{W}}^{(m,n)}]_* + \frac{1}{2}(\bar{\mathcal{V}}^{(n)} * \partial_{t_m} \mathcal{V} - \bar{\mathcal{V}}^{(m)} * \partial_{t_n} \mathcal{V}) \tag{2.21}$$

where

$$\begin{aligned} \bar{\mathcal{W}}^{(m,n)} &= \frac{1}{2}(\bar{\mathcal{V}}^{(n)} * \mathcal{V}^{(m)} - \bar{\mathcal{V}}^{(m)} * \mathcal{V}^{(n)})_{< 0} \\ &= -\frac{1}{2} \sum_{j=1}^{\infty} \lambda^{-j} \sum_{k=1}^{n-m} V_{n+j-k} * V_{m+k} \quad \text{if } m < n. \end{aligned} \tag{2.22}$$

Of course, we can use (2.1) to eliminate the t -derivatives in the last two terms of (2.21). As the λ^{-1} part of (2.21) we find

$$\partial_{\theta_{m,n}} V_1 = -\frac{1}{2} \left[V_0, \sum_{k=m+1}^n V_{m+n+1-k} * V_k \right]_* \quad \text{if } m < n. \tag{2.23}$$

Theorem 2. *The integrability condition (2.11), with $\mathcal{W}^{(m,n)}$ defined in (2.18), holds as a consequence of (2.1) and (2.12). The flows given by (2.1) commute with the flows (2.12).*

Proof. Multiplication of (2.21) with λ^k and restriction to the non-negative part leads to

$$\partial_{\theta_{m,n}} \mathcal{V}^{(k)} = ([\mathcal{V}^{(k)}, \bar{\mathcal{W}}^{(m,n)}]_* + \frac{1}{2}(\bar{\mathcal{V}}^{(n)} * \partial_{t_m} \mathcal{V}^{(k)} - \bar{\mathcal{V}}^{(m)} * \partial_{t_n} \mathcal{V}^{(k)}))_{\geq 0}.$$

Application of ∂_{t_k} to (2.18) yields

$$\begin{aligned} \partial_{t_k} \mathcal{W}^{(m,n)} &= \frac{1}{2}(\partial_{t_m} \mathcal{V}^{(k)} * \mathcal{V}^{(n)} - \partial_{t_n} \mathcal{V}^{(k)} * \mathcal{V}^{(m)}) + [\mathcal{V}^{(k)}, \mathcal{W}^{(m,n)}]_* \\ &\quad + ([\mathcal{V}^{(k)}, \bar{\mathcal{W}}^{(m,n)}]_* + \frac{1}{2}(\bar{\mathcal{V}}^{(n)} * \partial_{t_m} \mathcal{V}^{(k)} - \bar{\mathcal{V}}^{(m)} * \partial_{t_n} \mathcal{V}^{(k)}))_{\geq 0}. \end{aligned}$$

Here we used (2.7) in the form

$$\partial_{t_k} \mathcal{V}^{(n)} = \partial_{t_n} \mathcal{V}^{(k)} + [\mathcal{V}^{(k)}, \mathcal{V}^{(n)}]_*$$

which with the help of (2.1) implies

$$\partial_{t_k} \bar{\mathcal{V}}^{(n)} = -\partial_{t_n} \mathcal{V}^{(k)} + [\mathcal{V}^{(k)}, \bar{\mathcal{V}}^{(n)}]_*.$$

Now we obtain

$$\partial_{t_k} \mathcal{W}^{(m,n)} - \partial_{\theta_{m,n}} \mathcal{V}^{(k)} = [\mathcal{V}^{(k)}, \mathcal{W}^{(m,n)}]_* + \frac{1}{2}(\partial_{t_m} \mathcal{V}^{(k)} * \mathcal{V}^{(n)} - \partial_{t_n} \mathcal{V}^{(k)} * \mathcal{V}^{(m)}).$$

This is the integrability condition (2.11) which in fact implies the commutativity of the t - and θ -flows. □

Theorem 3. *The integrability condition (2.13), with $\mathcal{W}^{(m,n)}$ defined in (2.18), holds as a consequence of (2.1) and (2.12). The θ -flows given by (2.12) commute for all $m, n \geq 0$.*

Proof. This is a straightforward but tedious calculation. Let us write $A^{(m,n,k,l)}$ for the expression on the right-hand side of (2.13). First we derive

$$\begin{aligned} \partial_{\theta_{k,l}} \mathcal{W}^{(m,n)} &= \frac{1}{2}(\partial_{t_m} \mathcal{W}^{(k,l)} * \mathcal{V}^{(n)} - \partial_{t_n} \mathcal{W}^{(k,l)} * \mathcal{V}^{(m)} + \bar{\mathcal{V}}^{(n)} * \partial_{t_m} \mathcal{W}^{(k,l)} - \bar{\mathcal{V}}^{(m)} * \partial_{t_n} \mathcal{W}^{(k,l)} \\ &\quad - \partial_{t_k} \tilde{\mathcal{W}}^{(m,n)} * \mathcal{V}^{(l)} + \partial_{t_l} \tilde{\mathcal{W}}^{(m,n)} * \mathcal{V}^{(k)} - 2[\tilde{\mathcal{W}}^{(m,n)}, \mathcal{W}^{(k,l)}]_* + \frac{1}{2}(\partial_{t_k} \bar{\mathcal{V}}^{(n)} * \partial_{t_m} \mathcal{V}^{(l)} \\ &\quad - \partial_{t_l} \bar{\mathcal{V}}^{(n)} * \partial_{t_m} \mathcal{V}^{(k)} - \partial_{t_k} \bar{\mathcal{V}}^{(m)} * \partial_{t_n} \mathcal{V}^{(l)} + \partial_{t_l} \bar{\mathcal{V}}^{(m)} * \partial_{t_n} \mathcal{V}^{(k)}))_{\geq 0} \end{aligned}$$

where $\tilde{\mathcal{W}}^{(m,n)} = \mathcal{W}^{(m,n)} + \bar{\mathcal{W}}^{(m,n)}$. Here we used the expression for $\partial_{\theta_{k,l}} \mathcal{V}^{(n)}$ obtained in the proof of theorem 2 and a corresponding expression for $\partial_{\theta_{k,l}} \bar{\mathcal{V}}^{(n)}$. Then we applied (2.7) and (2.11) (which holds according to theorem 2) several times. With the help of the above expression, we find

$$\begin{aligned} A^{(m,n,k,l)} &= \frac{1}{2}(-2[\tilde{\mathcal{W}}^{(m,n)}, \bar{\mathcal{W}}^{(k,l)}]_* + \bar{\mathcal{V}}^{(n)} * \partial_{t_m} \tilde{\mathcal{W}}^{(k,l)} - \bar{\mathcal{V}}^{(m)} * \partial_{t_n} \tilde{\mathcal{W}}^{(k,l)} \\ &\quad - \bar{\mathcal{V}}^{(l)} * \partial_{t_k} \tilde{\mathcal{W}}^{(m,n)} + \bar{\mathcal{V}}^{(k)} * \partial_{t_l} \tilde{\mathcal{W}}^{(m,n)} + \partial_{t_m} \tilde{\mathcal{W}}^{(k,l)} * \mathcal{V}^{(n)} - \partial_{t_n} \tilde{\mathcal{W}}^{(k,l)} * \mathcal{V}^{(m)} \\ &\quad - \partial_{t_k} \tilde{\mathcal{W}}^{(m,n)} * \mathcal{V}^{(l)} + \partial_{t_l} \tilde{\mathcal{W}}^{(m,n)} * \mathcal{V}^{(k)} + \frac{1}{2}(\partial_{t_k} \bar{\mathcal{V}}^{(n)} * \partial_{t_m} \mathcal{V}^{(l)} - \partial_{t_l} \bar{\mathcal{V}}^{(n)} * \partial_{t_m} \mathcal{V}^{(k)} \\ &\quad - \partial_{t_k} \bar{\mathcal{V}}^{(m)} * \partial_{t_n} \mathcal{V}^{(l)} + \partial_{t_l} \bar{\mathcal{V}}^{(m)} * \partial_{t_n} \mathcal{V}^{(k)} - \partial_{t_m} \bar{\mathcal{V}}^{(l)} * \partial_{t_k} \mathcal{V}^{(n)} + \partial_{t_n} \bar{\mathcal{V}}^{(l)} * \partial_{t_k} \mathcal{V}^{(m)} \\ &\quad + \partial_{t_m} \bar{\mathcal{V}}^{(k)} * \partial_{t_l} \mathcal{V}^{(n)} - \partial_{t_n} \bar{\mathcal{V}}^{(k)} * \partial_{t_l} \mathcal{V}^{(m)}))_{\geq 0}. \end{aligned}$$

Inserting $\tilde{\mathcal{W}}^{(m,n)} = \frac{1}{2}(\bar{\mathcal{V}}^{(n)} * \mathcal{V}^{(m)} - \bar{\mathcal{V}}^{(m)} * \mathcal{V}^{(n)})$ and using (2.7) in various forms, a lengthy calculation leads to $A^{(m,n,k,l)} = 0$, so that (2.13) is indeed satisfied. This condition implies the commutativity of the deformation flows. □

We have thus shown that (2.12) defines a hierarchy of deformation equations which extends the noncommutative hierarchy (2.1) of Lax equations to a larger hierarchy.

3. The ncAKNS hierarchy and its reductions

Let us look at the compatibility condition of the second member ($n = 1$) of the Lax hierarchy (2.1) with the n th member ($n > 1$). Setting $t_1 = x$ and using the notation used in the introduction, we deal with

$$U = \mathcal{V}^{(1)} \quad \mathcal{V} = \mathcal{V}^{(n)}. \tag{3.1}$$

Furthermore, we set

$$V_0 = H \quad V_1 = U \tag{3.2}$$

where H is (in accordance with the first of equations (2.6)) a constant $N \times N$ matrix, which splits the algebra Mat_N of $N \times N$ matrices in such a way that $\text{Mat}_N = \ker(\text{ad}H) \oplus \text{Im}(\text{ad}H)$ (see also [31, 32]). Every matrix then has a unique decomposition $M = M^{(+)} + M^{(-)}$ where $M^{(+)} \in \ker(\text{ad}H)$ and $M^{(-)} \in \text{Im}(\text{ad}H)$. For the corresponding parts of V_k we write

$$A_k = V_k^{(-)} \quad B_k = V_k^{(+)}. \tag{3.3}$$

In the following we use lower indices x, t_n to indicate derivatives of functions or matrices with respect to these variables. For example, $U_x = \partial_x U$.

Inserting our ansatz in (1.3), which is a special case of (2.7), we obtain

$$U_x = [H, V_2] \tag{3.4}$$

$$V_{k,x} = [U, V_k]_* + [H, V_{k+1}] \quad k = 2, \dots, n - 1, \tag{3.5}$$

$$U_{t_n} = V_{n,x} - [U, V_n]_*. \tag{3.6}$$

Considering the family of equations with $n \geq 0$, we may write the last equation as

$$U_{t_n} = [H, V_{n+1}]_* \tag{3.7}$$

(see also [31]). The above equations are precisely those which we obtain from the Lax hierarchy (2.6).⁶ Equation (3.4) suggests to choose $U \in \text{Im}(\text{ad}H)$, i.e.

$$U = U^{(-)}. \tag{3.8}$$

The above system of equations then decomposes into

$$U_x = [H, A_2] \tag{3.9}$$

$$A_{k,x} = [U, B_k]_* + [H, A_{k+1}] \quad k = 2, \dots, n - 1 \tag{3.10}$$

$$B_{k,x} = [U, A_k]_* \quad k = 2, \dots, n - 1 \tag{3.11}$$

$$B_{n,x} = [U, A_n]_* \tag{3.12}$$

$$U_{t_n} = [H, A_{n+1}]_*. \tag{3.13}$$

The first equation determines A_2 . Then B_2 can be obtained from the third equation⁷ and the second then yields A_3 . By iteration we find the A_k and B_k . The above last equation is a deformed evolution equation.

In the following, we specialize to

$$H = \frac{1}{2\alpha}\sigma \quad \text{with} \quad \sigma^2 = I \tag{3.14}$$

where $\alpha \neq 0$ is a constant and I denotes the $N \times N$ unit matrix. Algebraically, we are then dealing with a deformation of the AKNS hierarchy [25–27] (which was originally expressed in terms of 2×2 matrices, see below). Then

⁶ See also lemma (2.4) in [31].

⁷ Introducing a formal inverse of ∂_x , as is common in soliton analysis, we can always solve for the B_k . In the case under consideration, we checked up to $k = 7$ that there is in fact a ‘quasi-local’ solution B_k , so that the right-hand side of (3.11) can be written as a total x -derivative of $*$ -products of U and its x -derivatives.

$$A_k = \alpha(-A_{k-1,x} + [U, B_{k-1}]_*)\sigma \quad B_{k,x} = [U, A_k]_* \quad k = 2, 3, \dots \quad (3.15)$$

and

$$\alpha U_{i_n} = -A_{n+1}\sigma \quad n = 0, 1, \dots \quad (3.16)$$

Besides $A_0 = 0$ and $A_1 = U$, for the first values of n we find

$$A_2 = -\alpha U_x \sigma \quad (3.17)$$

$$A_3 = -\alpha^2(-U_{xx} + 2U^{*3}) \quad (3.18)$$

$$A_4 = -\alpha^3(U_{xxx} - 3(U^{*2} * U_x + U_x * U^{*2}))\sigma \quad (3.19)$$

$$A_5 = -\alpha^4(-U_{xxxx} + 4U^{*2} * U_{xx} + 4U_{xx} * U^{*2} + 2U * U_{xx} * U + 6U_x * U * U_x + 2U * U_x^{*2} + 2U_x^{*2} * U - 6U^{*5}) \quad (3.20)$$

$$A_6 = -\alpha^5(U_{xxxxx} - 5(U^{*2} * U_{xxx} + U_{xxx} * U^{*2}) - 10(U_x * U * U_{xx} + U_{xx} * U * U_x) - 5(U * U_x * U_{xx} + U * U_{xx} * U_x + U_x * U_{xx} * U + U_{xx} * U_x * U) - 10U_x^{*3} + 10(U_x * U^{*4} + U^{*4} * U_x + U^{*2} * U_x * U^{*2}))\sigma \quad (3.21)$$

$$A_7 = -\alpha^6(-U_{xxxxxx} + 2U * U_{xxxx} * U + 6(U^{*2} * U_{xxxx} + U_{xxxx} * U^{*2}) + 9(U * U_x * U_{xxx} + U_{xxx} * U_x * U) + 4(U * U_{xxx} * U_x + U_x * U_{xxx} * U) + 15(U_x * U * U_{xxx} + U_{xxx} * U * U_x) + 11(U * U_{xx}^{*2} + U_{xx}^{*2} * U) + 20U_{xx} * U * U_{xx} + 20U_x * U_{xx} * U_x - 15(U^{*4} * U_{xx} + U_{xx} * U^{*4}) + 25(U_{xx} * U_x^{*2} + U_x^{*2} * U_{xx}) - 10(U^{*3} * U_{xx} * U + 2U^{*2} * U_{xx} * U^{*2} + U * U_{xx} * U^{*3}) - 10(U^{*3} * U_x^{*2} + U_x^{*2} * U^{*3}) - 15(U^{*2} * U_x^{*2} * U + U * U_x^{*2} * U^{*2}) - 25(U^{*2} * U_x * U * U_x + U_x * U * U_x * U^{*2}) - 5(U * U_x * U^{*2} * U_x + 2U * U_x * U * U_x * U + U_x * U^{*2} * U_x * U) - 20U_x * U^{*3} * U_x + 20U^{*7}) \quad (3.22)$$

$$A_8 = -7\alpha^7(U_{xxxxxxx}/7 - U_{xxxxx} * U^{*2} - U^{*2} * U_{xxxxx} - U * U_{xxxx} * U_x - U_x * U_{xxxx} * U - 2(U_{xxxx} * U_x * U + U * U_x * U_{xxxx}) - 3(U_{xxxx} * U * U_x + U_x * U * U_{xxxx}) - 5(U_{xxx} * U * U_{xx} + U_{xx} * U * U_{xxx}) - 3(U_{xxx} * U_{xx} * U + U * U_{xx} * U_{xxx}) - 2(U_{xx} * U_{xxx} * U + U * U_{xxx} * U_{xx}) - 4U_x * U_{xxx} * U_x - 7(U_{xxx} * U_x^{*2} + U_x^{*2} * U_{xxx}) + 4U^{*2} * U_{xxx} * U^{*2} + 3(U_{xxx} * U^{*4} + U^{*4} * U_{xxx}) - 8(U_{xx}^{*2} * U_x + U_x * U_{xx}^{*2}) - 10U_{xx} * U_x * U_{xx} + 4(U_{xx} * U_x * U^{*3} + U^{*3} * U_x * U_{xx}) + 4(U^{*2} * U_{xx} * U_x * U + U * U_x * U_{xx} * U^{*2}) + U_x * U_{xx} * U^{*3} + U^{*3} * U_{xx} * U_x) + 8(U_x * U * U_{xx} * U^{*2} + U^{*2} * U_{xx} * U * U_x) + 7(U_{xx} * U * U_x * U^{*2} + U^{*2} * U_x * U * U_{xx}) + 5(U_{xx} * U^{*3} * U_x + U_x * U^{*3} * U_{xx}) + 3(U_x * U^{*2} * U_{xx} * U + U * U_{xx} * U^{*2} * U_x + U^{*2} * U_x * U_{xx} * U + U * U_{xx} * U_x * U^{*2}) + 2(U_{xx} * U^{*2} * U_x * U + U * U_x * U^{*2} * U_{xx}) + 9(U_x^{*3} * U^{*2} + U^{*2} * U_x^{*3}) + 6(U * U_x^{*2} * U * U_x + U_x * U * U_x^{*2} * U) + 10U_x * U * U_x * U * U_x + 5(U_x^{*2} * U^{*2} * U_x + U_x * U^{*2} * U_x^{*2}) + 4(U_x^{*2} * U * U_x * U + U * U_x * U * U_x^{*2}) + 2U * U_x^{*3} * U - 5(U^{*6} * U_x + U_x * U^{*6} + U^{*4} * U_x * U^{*2} + U^{*2} * U_x * U^{*4}))\sigma \quad (3.23)$$

using the notation $U^{*2} = U * U$. Since σ is a constant matrix, the cyclic property of the trace requires $\text{Tr}(A_k \sigma) = 0$, and thus in particular $\text{Tr}(U \sigma) = 0$. Neglecting constants of integration, besides $B_0 = \sigma/2$ and $B_1 = 0$ we find

$$B_2 = -\alpha U^{*2} \sigma \tag{3.24}$$

$$B_3 = \alpha^2 [U, U_x]_* \tag{3.25}$$

$$B_4 = -\alpha^3 (U * U_{xx} + U_{xx} * U - U_x^{*2} - 3U^{*4}) \sigma \tag{3.26}$$

$$B_5 = -\alpha^4 ([U_{xxx}, U]_* - [U_{xx}, U_x]_* + 4U^{*2} * [U, U_x]_* + 4[U, U_x]_* * U^{*2} + 2U * [U, U_x]_* * U) \tag{3.27}$$

$$B_6 = -\alpha^5 (U_{xxxx} * U + U * U_{xxxx} - U_{xxx} * U_x - U_x * U_{xxx} + U_{xx}^{*2} - 5(U^{*3} * U_{xx} + U^{*2} * U_{xx} * U + U * U_{xx} * U^{*2} + U_{xx} * U^{*3}) + 5(U_x * U^{*2} * U_x - U * U_x * U * U_x - U_x * U * U_x * U - U * U_x^{*2} * U) + 10U^{*6}) \sigma \tag{3.28}$$

$$B_7 = -\alpha^6 ([U_{xxxx}, U]_* + [U_x, U_{xxxx}]_* + [U_{xxx}, U_{xx}]_* + 6[U^{*3}, U_{xxx}]_* + 4U * [U_{xxx}, U]_* * U + 3(U^{*2} * U_x * U_{xx} - U_{xx} * U_x * U^{*2}) + 9(U * U_x * U * U_{xx} - U_{xx} * U * U_x * U) + 6(U_{xx} * U^{*2} * U_x - U_x * U^{*2} * U_{xx}) + 8(U^{*2} * U_{xx} * U_x - U_x * U_{xx} * U^{*2}) + U * [U_{xx}, U_x]_* * U + 11(U * U_{xx} * U * U_x - U_x * U * U_{xx} * U) + 13[U, U_x^{*3}]_* + 3(U_x * U * U_x^{*2} - U_x^{*2} * U * U_x) + 15[U_x, U^{*5}]_* + 5(U^{*4} * U_x * U - U * U_x * U^{*4}) + 10U^{*2} * [U_x, U]_* * U^{*2}). \tag{3.29}$$

Let us elaborate the SW deformation equations which are given by (2.23). We restrict our considerations to the deformations between the second ($n = 1$) and the remaining ncAKNS equations ($n > 1$) and write

$$\theta_n = \theta_{1,n} \quad n = 0, 2, \dots \tag{3.30}$$

We obtain

$$U_{\theta_n} = \frac{1}{2\alpha} \left(\sum_{k=2}^n V_{n+2-k} * V_k \right)^{(-)} \sigma = \frac{1}{2\alpha} \sum_{k=2}^n (A_k * B_{n+2-k} + B_k * A_{n+2-k}) \sigma \tag{3.31}$$

which determines SW deformation equations for the members of the hierarchy. The first equations, corresponding to $n = 0, 2, 3, 4, 5, 6$, are

$$U_{\theta_0} = 0 \tag{3.32}$$

$$U_{\theta_2} = \frac{\alpha}{2} [U_x, U^{*2}]_* \sigma \tag{3.33}$$

$$U_{\theta_3} = \frac{\alpha^2}{2} ([U^{*2}, U_{xx}]_* + [U_x^{*2}, U]_*) \tag{3.34}$$

$$U_{\theta_4} = \frac{\alpha^3}{2} ([U_{xxx}, U^{*2}]_* + [U_x, U_{xx}]_* * U + U * [U_x, U_{xx}]_* + 4[U^{*4}, U_x]_* + 2U * [U^{*2}, *U_x] * U) \sigma \tag{3.35}$$

$$U_{\theta_5} = \frac{\alpha^4}{2} ([U^{*2}, U_{xxxx}]_* + [U_x * U_{xxx} + U_{xxx} * U_x - U_{xx}^{*2}, U]_* + 5[U_{xx}, U^{*4}]_* + 5(U * [U_x^{*2}, U]_* * U + [U_x * U, U_x * U^{*2}]_* + [U * U_x, U^{*2} * U_x]_*)) \tag{3.36}$$

$$\begin{aligned}
U_{\theta_6} = & \frac{\alpha^5}{2} ([U_{xxxxx}, U^{*2}]_* + U * [U_x, U_{xxxx}]_* + [U_x, U_{xxxx}]_* * U + U * [U_{xxx}, U_{xx}]_* \\
& + [U_{xxx}, U_{xx}]_* * U + 6[U^{*4}, U_{xxx}]_* + 2U * [U^{*2}, U_{xxx}]_* * U \\
& + 9(U^{*2} * U_x * U * U_{xx} + U^{*2} * U_{xx} * U_x * U - U * U_x * U_{xx} * U^{*2} \\
& - U_{xx} * U * U_x * U^{*2}) + 6(U * U_{xx} * U^{*2} * U_x + U_{xx} * U^{*2} * U_x * U \\
& - U * U_x * U^{*2} * U_{xx} - U_x * U^{*2} * U_{xx} * U) + 3(U^{*3} * U_x * U_{xx} \\
& - U_{xx} * U_x * U^{*3}) + 8(U^{*3} * U_{xx} * U_x - U_x * U_{xx} * U^{*3}) \\
& + 2(U^{*2} * U_x * U_{xx} * U - U * U_{xx} * U_x * U^{*2} + U * U_{xx} * U * U_x * U \\
& - U * U_x * U * U_{xx} * U) + 11(U^{*2} * U_{xx} * U * U_x - U_x * U * U_{xx} * U^{*2}) \\
& + 13[U^{*2}, U_x^{*3}]_* + 3[U * U_x * U, U_x^{*2}]_* + 3[U_x, U * U_x^{*2} * U]_* \\
& + 5(U^{*2} * [U_x, U^{*2}]_* * U^{*2} + 2U * [U_x, U^{*4}]_* * U + 3[U_x, U^{*6}]_*) \sigma \quad (3.37)
\end{aligned}$$

$$\begin{aligned}
U_{\theta_7} = & \frac{\alpha^6}{2} ([U^{*2}, U_{xxxxxx}]_* + [U_x * U_{xxxxx} + U_{xxxxx} * U_x, U]_* + [U, U_{xx} * U_{xxxx} \\
& + U_{xxxx} * U_{xx}]_* + 7[U_{xxxx}, U^{*4}]_* + [U_{xxxx}^{*2}, U]_* + 7(U_{xxx} * U_x * U^{*3} \\
& - U^{*3} * U_x * U_{xxx} + 2(U_{xxx} * U * U_x * U^{*2} - U^{*2} * U_x * U * U_{xxx}) \\
& + U * U_x * U^{*2} * U_{xxx} - U_{xxx} * U^{*2} * U_x * U + U * U_{xxx} * U_x * U^{*2} \\
& - U^{*2} * U_x * U_{xxx} * U + U * U_{xxx} * U^{*2} * U_x - U_x * U^{*2} * U_{xxx} * U \\
& + 2(U_x * U * U_{xxx} * U^{*2} - U^{*2} * U_{xxx} * U * U_x + U * U_x * U_{xxx} * U^{*2} \\
& - U^{*2} * U_{xxx} * U_x * U) + 2[U_{xx}^{*2}, U^{*3}]_* + 3[U_{xx} * U * U_{xx}, U^{*2}]_* \\
& + U * [U_{xx}^{*2}, U]_* * U + [U_{xx} * U^{*2} * U_{xx}, U]_* + 4(U_{xx} * U_x^{*2} * U^{*2} \\
& - U^{*2} * U_x^{*2} * U_{xx}) + U * U_x^{*2} * U * U_{xx} - U_{xx} * U * U_x^{*2} * U \\
& + 2[U_x * U_{xx} * U_x, U^{*2}]_* + 2(U * U_x * U * U_{xx} * U_x \\
& - U_x * U_{xx} * U * U_x * U) + U * U_x^{*2} * U_{xx} * U - U * U_{xx} * U_x^{*2} * U \\
& + U * U_{xx} * U * U_x^{*2} - U_x^{*2} * U * U_{xx} * U + 3(U_x^{*2} * U_{xx} * U^{*2} \\
& - U^{*2} * U_{xx} * U_x^{*2}) + 2(U * U_x * U_{xx} * U * U_x - U_x * U * U_{xx} * U_x * U) \\
& + 3[U^{*6}, U_{xx}]_* + U^{*2} * [U^{*2}, U_{xx}]_* U^{*2} + 3[U, U_x^{*4}]_* + [U^{*5}, U_x^{*2}]_* \\
& + 4[U^{*4}, U_x * U * U_x]_* + [U_x * U^{*2} * U_x, U^{*3}]_* + 2[U^{*2}, U_x * U^{*3} * U_x]_* \\
& + U^{*2} * [U_x^{*2}, U]_* * U^{*2} + 3[U_x * U^{*4} * U_x, U]_* + 3U * [U^{*3}, U_x^{*2}]_* * U \\
& + 3U * [U, U_x * U^{*2} * U_x]_* * U). \quad (3.38)
\end{aligned}$$

For each $n = 2, 3, \dots$, equation (3.16) determines a deformed evolution equation and (3.31) yields the associated SW deformation equation. Using the computer algebra program FORM [33, 34] we checked the commutativity of the ncAKNS flows and their SW deformation flows in the cases $n = 2, \dots, 7$. In section 2, we provided the corresponding general proofs.

Choosing 2×2 matrices (hence $N = 2$) and for σ the Pauli matrix

$$\sigma_3 = \text{diag}(1, -1) \quad (3.39)$$

we are dealing with the original AKNS hierarchy [25–27]. We have extended it to a ‘noncommutative space’ by introducing a Moyal-deformation with respect to the coordinates involved and derived the corresponding SW deformation equations. Several integrable models including the KdV, mKdV and NLS equations are obtained as reductions of the AKNS

hierarchy [25–27]. The corresponding deformed soliton equations and hierarchies result from the above deformed AKNS hierarchy. In the following we list some examples and derive from (3.31) the corresponding SW deformation equations.

(1) For even n , let us choose $\alpha = -i$ and

$$U = \begin{pmatrix} 0 & q \\ \pm \bar{q} & 0 \end{pmatrix} \tag{3.40}$$

where \bar{q} denotes the complex conjugate of the function q . We assume $\bar{\theta} = \theta$ in this case. From (3.16) we obtain the hierarchy

$$iq_{t_2} = -q_{xx} \pm 2q * \bar{q} * q \tag{3.41}$$

$$iq_{t_4} = -(q_{xxxx} \mp 4q * \bar{q} * q_{xx} \mp 2q * \bar{q}_{xx} * q \mp 4q_{xx} * \bar{q} * q \mp 2q * \bar{q}_x * q_x \mp 6q_x * \bar{q} * q_x \mp 2q_x * \bar{q}_x * q + 6q * \bar{q} * q * \bar{q} * q) \tag{3.42}$$

⋮

where the first equation is the ncNLS equation [9, 10], the deformation of the NLS equation. The corresponding SW deformation hierarchy begins with

$$q_{\theta_2} = \pm \frac{i}{2}(q_x * \bar{q} * q - q * \bar{q} * q_x) \tag{3.43}$$

$$q_{\theta_4} = \pm \frac{i}{2}[q_{xxx} * \bar{q} * q - q * \bar{q} * q_{xxx} + (q_x * \bar{q}_{xx} - q_{xx} * \bar{q}_x) * q + q * (\bar{q}_x * q_{xx} - \bar{q}_{xx} * q_x)] + i[2(q * \bar{q})^{*2} * q_x - 2q_x * (\bar{q} * q)^{*2} + q * (\bar{q} * q * \bar{q}_x - \bar{q}_x * q * \bar{q}) * q]. \tag{3.44}$$

For $n = 6$ already rather lengthy expressions arise which we omit here. They can be quite easily generated from our general formulae. Equation (3.43) already appeared in [9].

(2) For odd n , $\alpha = 1$ and

$$U = \begin{pmatrix} 0 & u \\ 1 & 0 \end{pmatrix} \tag{3.45}$$

we obtain the hierarchy⁸

$$u_{t_3} = (u_{xx} - 3u^{*2})_x \tag{3.46}$$

$$u_{t_5} = (u_{xxxx} - 5(u_{xx} * u + u * u_{xx} + u_x^{*2}) + 10u^{*3})_x \tag{3.47}$$

$$u_{t_7} = (u_{xxxxxx} - 7(u_{xxxx} * u + u * u_{xxxx}) - 14(u_{xxx} * u_x + u_x * u_{xxx}) - 21u_{xx}^2 + 21(u_{xx} * u^{*2} + u^{*2} * u_{xx}) + 28u * u_{xx} * u + 14(u_x * u * u_x + 2 * u * u_x^{*2} + 2u_x^{*2} * u) + 35u^{*4})_x \tag{3.48}$$

⋮

The first equation is the ncKdV equation [5, 11], the deformation of the KdV equation. The corresponding SW deformation hierarchy starts with

⁸ The fact that the right-hand sides can be written as total x -derivatives, so that these equations are conservation laws with common conserved density u , is a special feature of this reduction.

$$u_{\theta_3} = \frac{1}{2}[u, u_{xx}]_* = \frac{1}{2}([u, u_x]_*)_x \tag{3.49}$$

$$u_{\theta_5} = \frac{1}{2}([u, u_{xxx}]_* + [u_{xx}, u_x]_* + 5[u_x, u^{*2}]_*)_x \tag{3.50}$$

$$u_{\theta_7} = \frac{1}{2}([u, u_{xxxx}]_* + [u_{xxx}, u_x]_* + [u_{xx}, u_{xxx}]_* + 7([u_{xxx}, u^{*2}]_* + [(u_x^{*2})_x, u]_* + [u_x, u * u_{xx} + u_{xx} * u]_* + u * [u, u_x]_* * u) + 21[u^{*3}, u_x]_*)_x \tag{3.51}$$

The first equation was obtained in a different way in [5] where it has been used to calculate nc-corrections to the two-soliton solution of the KdV equation.

(3) We restrict again to odd n , but this time we choose

$$U = \begin{pmatrix} 0 & v \\ v & 0 \end{pmatrix} \tag{3.52}$$

which leads to the ncmKdV hierarchy

$$v_{t_3} = v_{xxx} - 3(v * v * v_x + v_x * v * v) \tag{3.53}$$

$$v_{t_5} = -(5(v_{xxx} * v^{*2} + v^{*2} * v_{xxx}) + 5(v_{xx} * v_x * v + v * v_x * v_{xx}) + v_x * v_{xx} * v + v * v_{xx} * v_x + 2v_{xx} * v * v_x + 2v_x * v * v_{xx}) + 10v_x^{*3} - 10(v_x * v^{*4} + v^{*2} * v_x * v^{*2} + v^{*4} * v_x) - v^{*5}) \tag{3.54}$$

⋮

The corresponding SW deformation hierarchy starts with

$$v_{\theta_3} = \frac{1}{2}([v * v, v_{xx}]_* - [v, v_x * v_x]_*) \tag{3.55}$$

$$v_{\theta_5} = \frac{1}{2}([v^{*2}, v_{xxx}]_* + [v_x * v_{xxx} + v_{xxx} * v_x, v]_*) + [v, v_{xx}^{*2}]_* + 5[v_{xx}, v^{*4}]_* + 5v * [v_x^{*2}, v]_* * v + 5[v * v_x, v^{*2} * v_x]_* + 5[v_x * v, v_x * v^{*2}]_* \tag{3.56}$$

The $n = 1$ equations were obtained in a different way in [5].

Remarks.

- (1) For some fixed $n > 1$ we may set $\theta_{1,n} = 0$ with the effect that the $*$ -product does not contain derivatives with respect to the ‘time’ t_n . In this case we still have an ordinary evolution equation and the deformation only involves the ‘spatial’ coordinate x and the remaining t_m where $m \neq n$. Interpreting t_n as a physical time coordinate would mean that, with $\theta_{1,n} \neq 0$, we were dealing with a ‘spacetime’ deformation, which is rather speculative, though such deformations have been discussed in the context of string theories. This deformation turns the classical evolution equation into one which is non-local in time. The interpretation of any of the parameters t_n appearing in the above hierarchy as a ‘time’ in a physical sense is not obligatory, however.
- (2) More generally, we may consider deformations between each pair of levels $m, n \geq 0$, allowing $\theta_{m,n} \neq 0$. With the assumptions made in this section, the deformation equations (2.23) take the form

$$\partial_{\theta_{(m,n)}} U = \frac{1}{2\alpha} \sum_{k=1}^{n-m} (A_{n+1-k} * B_{m+k} + B_{n+1-k} * A_{m+k})\sigma \tag{3.57}$$

which have to be evaluated by inserting the above expressions for the A_k and B_k . For example, in the ncKdV case we find

$$u_{\theta_{3,5}} = \frac{1}{2}([u_{xx}, u_{xxx}]_* + 3[u_{xxx}, u^{*2}]_* + 2[(u^{*2})_x, u_{xx}]_* + 2(u * u_x * u_{xx} - u_{xx} * u_x * u) + 12u * [u, u_x]_* * u + 6[u^{*3}, u_x]_*)_x \tag{3.58}$$

(3) Without deformation, the AKNS hierarchy has common conserved densities given by $\text{Tr}(V_k \sigma) = \text{Tr}(B_k \sigma)$ [31]. Due to the noncommutativity of the $*$ -product, these are in general no longer expressions for the conserved densities of the deformed hierarchy.

4. The extended ncKP hierarchy

Let

$$L = \partial + \sum_{j=1}^{\infty} u_{j+1} \partial^{-j} \quad L^{(n)} = (L^n)_{\geq 0} \quad \bar{L}^{(n)} = (L^n)_{< 0} = L^n - L^{(n)} \quad (4.1)$$

where we simply write L^n instead of L^{*n} . The non-negative (negative) part of a formal series is now understood in the sense of non-negative (negative) powers of $\partial = \partial_x$. Using the derivation property of ∂ and

$$\partial^{-1} u_k = u_k \partial^{-1} - u_{k,x} \partial^{-2} + u_{k,xx} \partial^{-3} - u_{k,xxx} \partial^{-4} + \dots \quad (4.2)$$

we obtain

$$L^{(1)} = \partial \quad (4.3)$$

$$L^{(2)} = \partial^2 + 2u_2 \quad (4.4)$$

$$L^{(3)} = \partial^3 + 3u_2 \partial + 3(u_{2,x} + u_3) \quad (4.5)$$

$$L^{(4)} = \partial^4 + 4u_2 \partial^2 + (6u_{2,x} + 4u_3) \partial + 4u_{2,xx} + 6u_2^{*2} + 6u_{3,x} + 4u_4. \quad (4.6)$$

The *ncKP hierarchy* (see also [16]) is the set of equations

$$\partial_{t_n} L = [L^{(n)}, L]_* \quad n = 1, 2, \dots \quad (4.7)$$

where $t_1 = x$. This is a deformation of the classical KP hierarchy [28–30], using the $*$ -product given by (2.4). Elaborating these equations for $n = 2, 3$, we find

$$\partial_{t_2} u_2 = (u_{2,x} + 2u_3)_x \quad (4.8)$$

$$\partial_{t_2} u_3 = u_{3,xx} + 2u_{4,x} + 2u_2 * u_{2,x} + 2[u_2, u_3]_* \quad (4.9)$$

$$\partial_{t_2} u_4 = 2u_{5,x} + u_{4,xx} + 4u_3 * u_{2,x} - 2u_2 * u_{2,xx} + 2[u_2, u_4]_* \quad (4.10)$$

$$\partial_{t_2} u_5 = 2u_{6,x} + u_{5,xx} + 2u_2 * u_{2,xx} - 6u_3 * u_{2,xx} + 6u_4 * u_{2,x} + 2[u_2, u_5]_* \quad (4.11)$$

⋮

$$\partial_{t_3} u_2 = (u_{2,xx} + 3u_2^{*2} + 3u_{3,x} + 3u_4)_x \quad (4.12)$$

$$\partial_{t_3} u_3 = u_{3,xxx} + 3u_{4,xx} + 6u_2 * u_{3,x} + 3u_{2,x} * u_3 + 3u_3 * u_{2,x} + 3u_{5,x} + 3[u_2, u_4]_* \quad (4.13)$$

$$\begin{aligned} \partial_{t_3} u_4 = & u_{4,xxx} + 3u_{5,xx} - 3u_2 * u_{3,xx} - 3u_3 * u_{2,xx} + 6u_4 * u_{2,x} \\ & + 3(u_2 * u_4)_x + 3u_{6,x} + 3[u_3, u_4]_* + 3[u_2, u_5]_* \end{aligned} \quad (4.14)$$

$$\begin{aligned} \partial_{t_3} u_5 = & 3u_{7,x} + u_{5,xxx} + 3u_{6,xx} + 3u_3 * u_{2,xxx} + 3u_2 * u_{3,xxx} - 9u_3 * u_{3,xx} - 9u_4 * u_{2,xx} \\ & + 3(u_2 * u_5)_x + 9u_5 * u_{2,x} + 9u_4 * u_{3,x} + 3[u_2, u_6]_* + 3[u_3, u_5]_* \end{aligned} \quad (4.15)$$

⋮

(see also [16]). Eliminating u_3 and u_4 from (4.8), (4.9) and (4.12), one recovers a deformed version [11, 16] of the KP equation,

$$\left(4u_{2,t} - u_{2,xxx} - 6(u_2^{*2})_x + 6 \left[u_2, \int u_{2,y} dx \right]_* \right)_x - 3u_{2,yy} = 0 \quad (4.16)$$

where $t_2 = y$ and $t_3 = t$.

The ncKP equations imply $\partial_{t_n} L^m = [L^{(n)}, L^m]_*$ which, using

$$0 = [L^n, L^m]_* = [L^{(n)}, L^{(m)}]_* + [L^{(n)}, \bar{L}^{(m)}]_* + [\bar{L}^{(n)}, L^{(m)}]_* + [\bar{L}^{(n)}, \bar{L}^{(m)}]_* \tag{4.17}$$

leads to

$$\partial_{t_n} L^{(m)} - \partial_{t_m} L^{(n)} + [L^{(m)}, L^{(n)}]_* = 0 \tag{4.18}$$

and

$$\partial_{t_n} \bar{L}^{(m)} - \partial_{t_m} \bar{L}^{(n)} - [\bar{L}^{(m)}, \bar{L}^{(n)}]_* = 0 \tag{4.19}$$

which is equivalent to (4.18). As a consequence of this equation and the Jacobi identity, the flows given by (4.7) indeed commute. Using (4.7), (4.18) can also be written as

$$\partial_{t_n} \bar{L}^{(m)} + \partial_{t_m} L^{(n)} + [\bar{L}^{(m)}, L^{(n)}]_* = 0. \tag{4.20}$$

The n th member of the ncKP hierarchy arises as the integrability condition of the linear system

$$L * \psi = \lambda \psi \quad \partial_{t_n} \psi = L^{(n)} * \psi \tag{4.21}$$

assuming $\partial_{t_n} \lambda = 0$. The integrability condition for two different members of the linear system, with coordinates t_n and t_m , respectively, is precisely (4.18) which we have seen to be satisfied as a consequence of (4.7). Following the recipe of the introduction, we extend the above linear system by imposing deformation equations of the form

$$\partial_{\theta_{mn}} \psi = W^{(m,n)} * \psi \tag{4.22}$$

assuming $\partial_{\theta_{mn}} \lambda = 0$. This results in new integrability conditions. We find

$$\partial_{\theta_{mn}} L = [W^{(m,n)}, L]_* + \frac{1}{2}(\partial_{t_n} L * L^{(m)} - \partial_{t_m} L * L^{(n)}) \tag{4.23}$$

and, by induction and using (2.10),

$$\partial_{\theta_{mn}} L^r = [W^{(m,n)}, L^r]_* + \frac{1}{2}((\partial_{t_n} L^r) * L^{(m)} - (\partial_{t_m} L^r) * L^{(n)}). \tag{4.24}$$

Further integrability conditions are

$$\partial_{t_r} W^{(m,n)} - \partial_{\theta_{mn}} L^{(r)} + [W^{(m,n)}, L^{(r)}]_* = \frac{1}{2}(\partial_{t_m} L^{(r)} * L^{(n)} - \partial_{t_n} L^{(r)} * L^{(m)}) \tag{4.25}$$

and

$$0 = \partial_{\theta_{rs}} W^{(m,n)} - \partial_{\theta_{mn}} W^{(r,s)} + [W^{(m,n)}, W^{(r,s)}]_* - \frac{1}{2}(\partial_{t_m} W^{(r,s)} * L^{(n)} - \partial_{t_n} W^{(r,s)} * L^{(m)} - \partial_{t_r} W^{(m,n)} * L^{(s)} + \partial_{t_s} W^{(m,n)} * L^{(r)}). \tag{4.26}$$

Since $(\partial_{\theta_{mn}} L)_{\geq 0} = 0$, equation (4.23) implies

$$\begin{aligned} ([W^{(m,n)}, L]_{*})_{\geq 0} &= \frac{1}{2}(\partial_{t_m} L * L^{(n)} - \partial_{t_n} L * L^{(m)})_{\geq 0} \\ &= \frac{1}{2}([L^{(m)}, L]_* * L^n - [L^{(n)}, L]_* * L^m)_{\geq 0} \\ &= \frac{1}{2}([L^{(m)} * L^n, L]_* - [L^{(n)} * L^m, L]_{*})_{\geq 0} \\ &= \frac{1}{2}([-\bar{L}^{(m)} * L^n + \bar{L}^{(n)} * L^m, L]_{*})_{\geq 0} \\ &= \frac{1}{2}([-\bar{L}^{(m)} * L^{(n)} + \bar{L}^{(n)} * L^{(m)}, L]_{*})_{\geq 0} \end{aligned} \tag{4.27}$$

which suggests to set

$$W^{(m,n)} = \frac{1}{2}(\bar{L}^{(n)} * L^{(m)} - \bar{L}^{(m)} * L^{(n)})_{\geq 0}. \tag{4.28}$$

This expression is completely analogous to the corresponding ncAKNS expression (2.18). Introducing

$$\bar{W}^{(m,n)} = \frac{1}{2}(\bar{L}^{(n)} * L^{(m)} - \bar{L}^{(m)} * L^{(n)})_{< 0} \tag{4.29}$$

we also obtain

$$\partial_{\theta_{mn}} L^r = [L^r, \bar{W}^{(m,n)}]_* + \frac{1}{2}(\bar{L}^{(n)} * \partial_m L^r - \bar{L}^{(m)} * \partial_n L^r). \quad (4.30)$$

As a consequence of this equation and (4.7), the above integrability conditions are satisfied and it follows that the flows given by (4.23), with $W^{(m,n)}$ defined in (4.28), commute with each other and also commute with the flows given by (4.7). The corresponding calculations are rather tedious, however. We thus obtained an extension of the ncKP hierarchy.

In particular, from (4.30) we obtain

$$\partial_{\theta_{mn}} u_2 = \frac{1}{2} \partial_x (\bar{L}^{(n)} * L^{(m)} - \bar{L}^{(m)} * L^{(n)})_{-1} \quad (4.31)$$

so that, for example,

$$\partial_{\theta_{1,2}} u_2 = \frac{1}{2} (u_4 + u_{3,x} - u_2^{*2})_x = \frac{1}{6} (u_{2,t} - u_{2,xxx} - 6(u_2^{*2})_x). \quad (4.32)$$

Using FORM [33, 34] we checked that the flow determined by this equation indeed commutes with the ncKP flow.

5. Conclusions

We have shown that the noncommutative AKNS and KP hierarchies can be extended to larger hierarchies which contain the corresponding SW deformation equations. Of course, our procedure for generating deformation equations for which the flow commutes with the flow of the deformed equation one starts with (and thus a symmetry of the latter equation) can be applied to other deformed integrable models and the corresponding hierarchies. One can say that, as a consequence of the explicit dependence on deformation parameters, deformed soliton equations have additional symmetries. We have demonstrated that there is much more structure behind *deformed* soliton equations than has been revealed up to now.

As in the case of the ncKdV equation treated in [5], SW deformation equations can be used to construct solutions of the corresponding nc-soliton equation from solutions of its classical version.

Conserved densities have already been obtained for several noncommutative versions of soliton equations, including the ncNLS and ncKdV equations (see [5, 9, 10, 16], in particular). It turned out that these cannot be expressed in terms of the $*$ -product only. Another ‘generalized $*$ -product’ was needed, which is commutative, but non-associative. It has been introduced in [35] and frequently used in recent work on noncommutative field theories (see [36, 37], for example). Because of technical problems associated with the generalized $*$ -product, the question of *common* conserved densities of extended noncommutative soliton hierarchies has not found an answer yet. A better understanding of the properties of the generalized $*$ -product seems to be required.

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